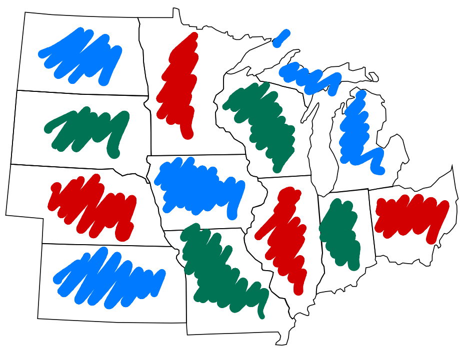
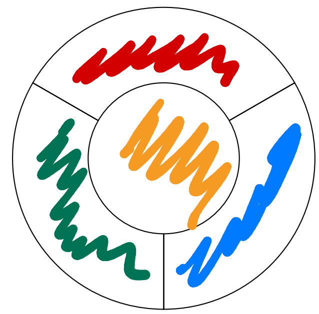
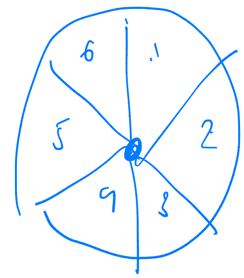
Lecture 5 - Coloring Planar Graphs

In 1852 Francis Guthrie observed that England region map can be colored with 4 colors such that any two regions sharing a border are colored differently. He observed that in general, one need at least 4 colors for coloring any map and came with the following problem.

1: Color states in Midwest and the silly map. Can you do Midwest with just three colors?







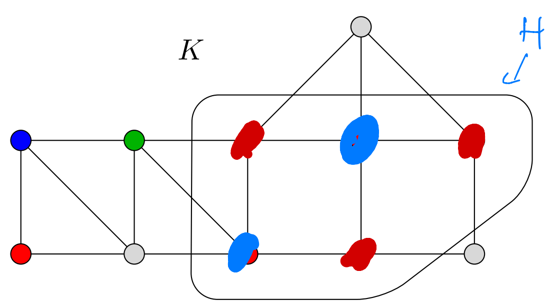
Four Color Problem (Francis Guthrie). *Regions of any planar map can be colored with four colors such that any two regions that share a common border are colored differently.*

Kempe chain: Let G be a graph, φ its proper coloring and c_1 and c_2 two colors. Let H be a maximal (in inclusion) connected subgraph of G such that for every $v \in V(H)$ holds $\varphi(v) \in \{c_1, c_2\}$. Define a coloring ϱ for every vertex $v \in V(G)$ in the following way:

$$\varrho(v) := \begin{cases} c_1 & \text{if } v \in V(H) \text{ and } \varphi(v) = c_2\,, \\ c_2 & \text{if } v \in V(H) \text{ and } \varphi(v) = c_1\,, \\ \varphi(v) & \text{otherwise.} \end{cases}$$

Due to the maximality of H, ϱ is a proper coloring. We call H a *Kempe chain.* Note that H is a component of the subgraph induced by c_1- and c_2-colored vertices. We usually denote this graph by $H(c_1, c_2)$.

2: Find a red-blue Kempe chain and switch the colors.



A classical example for Kempe chains is the proof of the Five Color Theorem.

Theorem 1 (Heawood)**.** *Every planar graph is 5-colorable.*

3: Prove the theorem using Kempe chains and degeneracy.

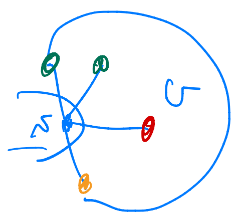
$\delta(G) \leq 5$

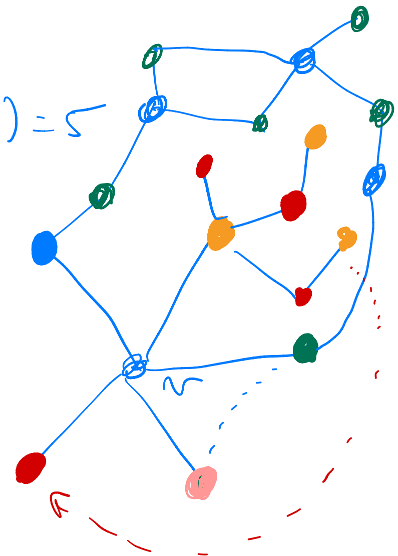
$d(v) = 5$

LUCKY IF 2 SAME

v OF MIN DEGREE

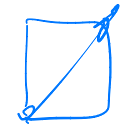
IF $d(v) \leq 4$ BY INDUCTION





5 colors up to used on neighbors

Theorem 2 (Heawood)**.** *A planar triangulation with every vertex of even degree is 3-colorable.*



Proof postponed to nowhere-zero integer flows class.

In 1969, Heesch presented the Discharging method and in 1977 using this method Appel and Haken succeeded in solving the Four Color Theorem using hard to verify proof. In 1995, Robertson, Sanders, Seymour and Thomas gave a new proof still based on computer assistant but a significantly shorter one.

Four Color Theorem (Appel and Haken). *Regions of any planar map can be colored with four colors such that any two regions that share a common border are colored differently.*

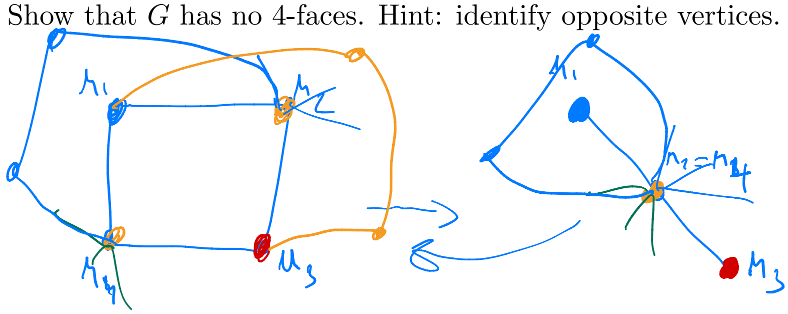
Theorem 3 (Grötzsch)**.** *Every planar traingle-free is 3-colorable.*

The original proof was technical. We present a short proof using by bound of Kostochka and Yancey. If G is a 4-critical graph on n vertices with m edges, then

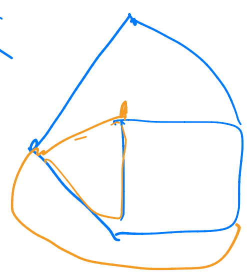
$$m \geq \frac{5n-2}{3}.$$

Proof. Let G be a minimal counterexample on n vertices, m edges, embedded in the plane with f faces. By the minimality, G is a 4-critical graph.

4: Show that G has no 4-faces. Hint: identify opposite vertices.



G

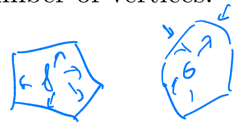


1 VERTEX LESS

NO 4-FACES $\quad 5f \leq \sum_{\text{FACE}} \ell(F) = 2m$

5: Give a lower bound on the number of edges using the number of vertices.

$5n - 5m + 5f = 2 \cdot 5$



$5f \leq 2m$

$5n - 3m \geq 10$

$\boxed{\frac{5n-10}{3} \geq m}$

6: Finish the proof.

$$\frac{5n-10}{3} \geq \frac{5n-2}{3}$$

□

1 Discharging method

Discharging is a very powerful technique for proving various theorems about planar graphs (almost anything about planar graphs).

Usually rather technical proofs, but great fit for the plane and local constraints like in coloring.

Discharging proof outline

- a minimum counterexample G
- List *reducible* configurations which cannot occur in G.
- Assign *initial charge* to vertices and faces of G.
- sum of all charges is negative (by Euler's formula).
- Apply some rules for shifting the charges between vertices and faces while preserving the total sum.
- Argue that if G has no reducible configuration, then the final charge of every face and every vertex is nonnegative (this gives a contradiction)

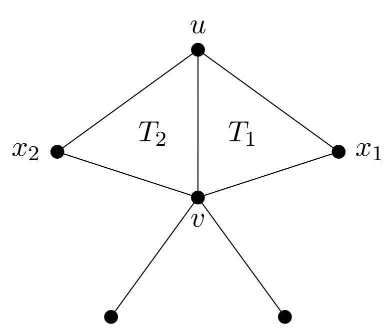
The proof gives that every planar graph contains at least ine of the reducible configurations.

Example:

Theorem 4. *Let G be a planar graph. Then $\chi(G) \leq 5$.*

Proof. Let G be a counterexample. We may assume that G is 6-critical, and as critical graphs are without clique-cuts, so G has no separating 3-cycle. It also gives minimum degree at least 5.

Reducible configurations. **7:** Show that *a 5-vertex v incident with two triangles T_1, T_2 sharing an edge containing v* is reducible.



Initial charges. By $\ell(f)$ we denote the length of a facial walk around f where bridges are counted twice. We define the initial charges ch for a vertex v and a face f in the following way:

$$\mathrm{ch}(v) = \deg(v) - 6 \quad \text{and} \quad \mathrm{ch}(f) = 2\ell(f) - 6.$$

8: Verify that the sum of all charges is negative by Euler's formula.

9: What as positive and what has negative charge? What are charges of vertices and faces with small degree or ℓ?

Discharging rules. We use only one discharging rule to redistribute the initial charge to increase the charge on 5-vertices. A face of size 4 or more, is a 4^+-*face*.

Rule 1. *Every* 4^+*-face sends charge* $\frac{1}{2}$ *to every adjacent 5-vertex.*

Final charges We use $\mathrm{ch}^*(x)$ to denote the final charge of a vertex or face x.

10: Show that $\mathrm{ch}^*(x) \geq 0$ for every vertex and face x.

□